Final exam: Math 232

December 11th, 8:30am–12:30pm

Instructions/Remarks:

- Read all instructions.
- The questions are on the 12 *numbered* single-sided pages following this one. Count them now.
- Please include all details and rough work. You may use the back of the question pages and there is extra paper should you require it.
- The exam is out of a total of 100 marks. The value of each question is indicated below.
- No calculators or electronic devices of any kind are permitted.
- You have 3 hours to complete this examination.
- Good luck!

Marks:

1) ______ /15 2) ______ /15 3) ______ /10 4) ______ /5 5) ______ /5

6) ______ /10 7) ______ /10 8) ______ /15 9) ______ /15

Total: ________ /100 Grade: ________

Name: ____________________ **SFU e-mail ID:** ____________________

Student Number: ______________________________

Signature: ______________________________

Question 1 [15pts] Suppose the entire population of a certain nation lives either in the city or the country. Consider the following model for how the population in the city and country changes year by year.

- Every year 10% of the people living in the city move to the country and the rest remain in the city.
- Every year 20% of the people living in the country move to the city and the rest remain in the country.

Let $x_1^{(k)}$ be the number of people living in the city in year k and let $x_2^{(k)}$ be the number of people living in the country in year k. Let $\mathbf{x}^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}$.

a. The model can be expressed as $\mathbf{x}^{(k+1)} = A\mathbf{x}^{(k)}$. What is A?

b. One of the eigenvalues of A is 1. What is the other eigenvalue?

c. Give an eigenvector corresponding to each eigenvalue of A.

d. Give an invertible matrix P and a diagonal matrix D, both of size 2×2 such that $AP = PD$.

(Continue work here)

Question 2 [15 pts] Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that reflects each point through the line $x = y$ and then stretches it in the x direction by a factor of 2.

a. What is the standard matrix for T?

b. What is the matrix for T relative to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$?

c. State a basis $\mathcal{C}$ for which the matrix for T relative to the basis $\mathcal{C}$ is diagonal.

(Continue work here)

Question 3 [10 pts] Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}.$$

a. Are these vectors linearly independent? Why or why not?

b. Find a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ that is a basis for $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$.

c. What is the dimension of $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$?

Question 4 [5 pts] Let $a, b, c, d, e, f, g, h, i$ be real numbers such that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 9.$$

State the following determinants. You do not need to justify your answer.

a. $\begin{vmatrix} d & e & f \\ g & h & i \\ a & b & c \end{vmatrix}$

b. $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$

c. $\begin{vmatrix} a & b & 0 \\ 3d & 3e & 0 \\ g & h & 0 \end{vmatrix}$

d. $\begin{vmatrix} a-2d & b-2e & c-2f \\ d & e & f \\ g & h & i \end{vmatrix}$

e. $\begin{vmatrix} 5a & 5b & 5c \\ d & e & f \\ g-a & h-b & i-c \end{vmatrix}$

Question 5 [5 pts] Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for $\mathbb{R}^3$. Let T be a linear transformation on $\mathbb{R}^3$ such that

$$T(\mathbf{b}_1) = \mathbf{b}_2 - \mathbf{b}_3, \quad T(\mathbf{b}_2) = \mathbf{b}_1 - \mathbf{b}_3, \quad T(\mathbf{b}_3) = \mathbf{b}_1 + \mathbf{b}_2.$$

a. What is $[T]_{\mathcal{B}}$, the matrix for T relative to the basis $\mathcal{B}$?

b. Is there a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$?

Question 6 [10 pts] Find a basis for the null space of each of the following matrices:

a. $\begin{bmatrix} 1 & 4 & 0 & -3 \\ 2 & 8 & 0 & -6 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Question 7 [10 pts] In each case, state whether the matrix A is invertible or non-invertible and briefly justify your answer.

a. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 5 & -3 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

c. A is a diagonalizable 4×4 matrix and has eigenvalues $1, 1, -i, i$.

d. A is 3×3 and $A^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$.

e. A is $n \times n$ and $\text{Col } A = \mathbb{R}^n$.

Question 8 [15 pts] Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

a. Compute an orthonormal basis for Col A.

b. Let $\hat{\mathbf{b}}$ be the orthogonal projection of the vector $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ onto Col A. Compute $\hat{\mathbf{b}}$.

c. Find a least-squares solution of system $A\mathbf{x} = \mathbf{b}$.

(Continue work here.)

Question 9 [15 pts] Suppose A has eigenvalues $\lambda = 0.9$ and $\lambda = 1.1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

a. Give diagonal D and invertible P such that $A = PDP^{-1}$.

b. Find a formula for A^{1000}. (Your answer should be a 2×2 matrix with entries of the form $C_0 a^{1000} + C_1 b^{1000}$ where C_0, C_1, a, b are constants.)

c. Let $x^{(k+1)} = Ax^{(k)}$ where $x^{(0)} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. For large k there is an approximate formula for $x^{(k)}$:

$$x^{(k)} \approx \gamma^k \begin{bmatrix} c \\ d \end{bmatrix},$$

where γ, c, d are constants. State γ, c, d.